Extension properties and separability of groups

A connection between model theory and profinite topologies

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September 23, 2025

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1. Introduction
2. Subgroup separability and EPPA for sets
3. Product separability and EPPA for structures
4. **V**-separability and the Ribes-Zalesskiĭ Theorem

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Theorem (Ribes-Zalesskiĭ 1993)

Free groups are 2-product separable: the product $H_1 H_2$ of any two finitely generated subgroups $H_1, H_2 \leq F$ is closed in the profinite topology of F.

Extension property for partial automorphisms

Throughout, *structures* are L-structures for a finite relational language L.

Definition (Herwig-Lascar 2000)

A class $\mathcal{C}$ of structures has the *extension property for partial automorphisms (EPPA)* if for each $M, M' \in \mathcal{C}$, with M finite and $M \leq M'$, and for each collection $p_1, \ldots, p_n$ of partial automorphisms of M extending to (total) automorphisms of M', there is a finite structure $N \in \mathcal{C}$ containing M as a substructure such that $p_1, \ldots, p_n$ extend to automorphisms of N.

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Theorem (Hrushovski 1991)

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Extensions of partial group actions

Definition (Coulbois 2001)

Let $M \in \mathcal{C}$ be a finite structure. A map $\varphi : G \to \mathrm{Part}(M)$ is a *partial action* if there is a finite symmetric subset $S \subseteq G$, an extension $M \leq M' \in \mathcal{C}$, and an action $\overline{\varphi} : G \to \mathrm{Aut}(M')$, such that for all $g \in G$ and $m_1, m_2 \in M$:

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Note that $\mathcal{C}$ has the EPPA $\Leftrightarrow$ all finitely generated free groups have the extension property for $\mathcal{C}$.

Connections between extension properties and separability

Theorem (Gitik 1997)

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Theorem (Herwig-Lascar 2000; Coulbois 2001)

A group is n-product separable iff it has the extension property for n-cycle free n-partitioned structures.

Other profinite topologies?

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Conjecture (Herwig-Lascar 2000)

Let $H \leq F$ be finitely generated. Then the following are equivalent.

1. *H is closed in the pro-odd topology (where $\mathbf{V}$ = odd-order finite groups).*
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The proof uses a strengthening of the Ribes-Zalesskiĭ theorem (free groups are product separable) to pro-$\mathbf{V}$ topologies:

Relative product separability

Theorem (Ribes-Zalesskiĭ 1994)

Let $\mathbf{V}$ be a pseudovariety of groups that is closed under extensions and let F be a free group. If $H_1, \ldots, H_n$ are finitely generated subgroups of F which are closed in the pro-$\mathbf{V}$ topology of F, then their product $H_1 \cdots H_n$ is also closed in the pro-$\mathbf{V}$ topology of F.

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Conjecture

A group G *is* 2*-product separable relative to* $\mathbf{V}$ *iff every partial action* φ *of* G *on a finite graph* M *with closed stabilizers extends to an action* $\widetilde{\varphi}$ *of* G *on a finite graph* $N \geq M$ *such that* $\widetilde{\varphi}(G)$ *is a pro-*$\mathbf{V}$ *group.*

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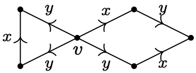

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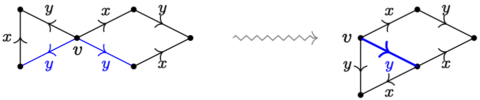

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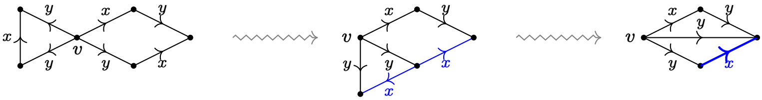

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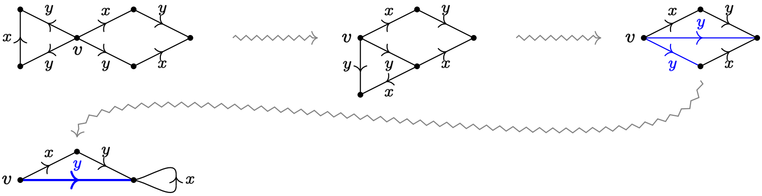

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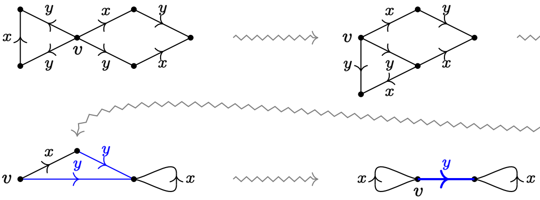

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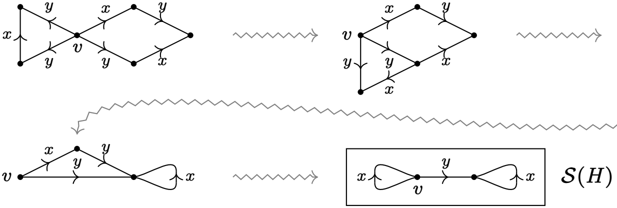

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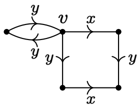

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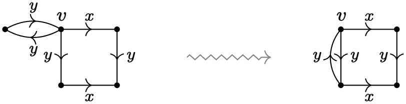

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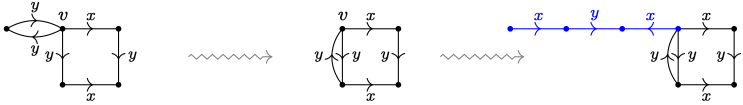

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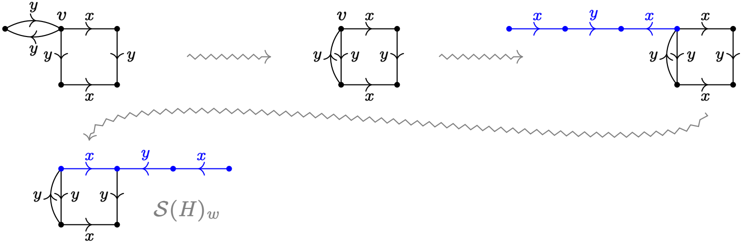

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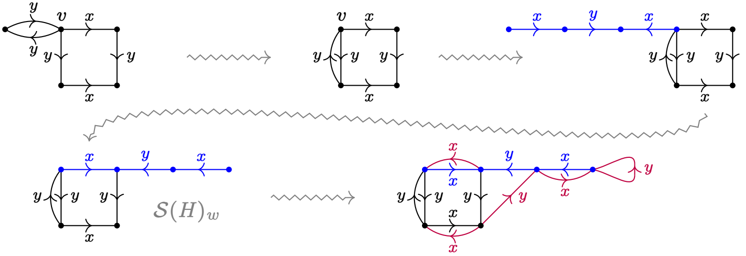
3. Expand $\mathcal{S}(H)_w$ to a cover of the rose. The deck transformation induced by w is nontrivial, so w is separated from H in the quotient.



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Proof sketch of (⇒).

Let $\varphi : G \to \mathrm{Part}(M)$ be a partial action, let $[c_1], \ldots, [c_m]$ be its orbits, and let $H_i \coloneqq \mathrm{Stab}_\varphi(c_i)$.

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In general, we let $N \coloneqq \bigsqcup_i G/K_i$, where $H_i \leq K_i \leq_{fi} G$ are obtained from 2-product separability of G and satisfy (1) and (2) for all a, a', b, b' above.

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Note that M is finite, so these K_i's can be obtained uniformly. ■

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For $i = 1, 2$, let $K_i := \mathrm{Stab}_\psi(H_i) \leq_{fi} G$. Then $H_i \leq K_i$, since if $h \in H_i$,

$$\psi(h)(H_i) = \varphi(s_1 \cdots s_l)(H_i) = (\overline{\varphi}(s_1) \circ \cdots \circ \overline{\varphi}(s_l))(H_i) = H_i$$

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where $h = s_1 \cdots s_l$ for some $s_j \in H$. Finally, $w \notin K_1 K_2$ since if $w = k_1 k_2$, we have $wk_2^{-1} = k_1$, so $\psi(wk_2^{-1})(H_1) = \psi(k_1)(H_1) = H_1$ and

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But then $M \models (H_1, wH_2)$, so $H_1 \cap wH_2 \neq \varnothing$, and hence $w \in H_1H_2$. ■

Contents

1 Introduction

2 Subgroup separability and EPPA for sets

3 Product separability and EPPA for structures

4 **V**-separability and the Ribes-Zalesskiĭ Theorem

The Ribes-Zalesskiĭ Theorem for pro-$\mathbf{V}$ topologies

Theorem (Ribes-Zalesskiĭ 1994)

Let $\mathbf{V}$ be a pseudovariety of groups that is closed under extensions and let F be a free group. If $H_1, \ldots, H_n$ are finitely generated subgroups of F which are closed in the pro-$\mathbf{V}$ topology of F, then their product $H_1 \cdots H_n$ is also closed in the pro-$\mathbf{V}$ topology of F.

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I will present a proof due to Auinger and Steinberg (2005).

Proof of the Ribes-Zalesskiĭ Theorem

Let $H_1, \ldots, H_n \leq F$ be finitely generated subgroups which are closed in the pro-$\mathbf{V}$ topology of F. We seek, for each word $w \in F$, a group $K \in \mathbf{V}$ such that if $[w]_K \in [H_1 \cdots H_n]_K$, then $w \in H_1 \cdots H_n$. Let $F = F_X$.

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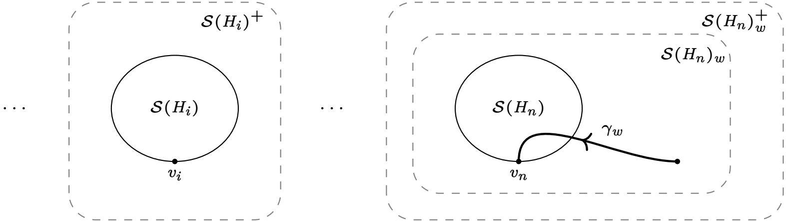
Motivated by the proof of Hall's Theorem, consider the group G of deck transformations of expansions of the Stallings' graphs $\Gamma_i \coloneqq (\mathcal{S}(H_i), v_i)$ for $1 \leq i < n$ and $\Gamma_n \coloneqq (\mathcal{S}(H_n)_w, v_n)$:

$$G \coloneqq \langle (f_x^1, \ldots, f_x^n) : x \in X \rangle \leq G_1 \times \cdots \times G_n.$$



Proof of the Ribes-Zalesskiĭ Theorem

Lemma

If $H \leq F$ is finitely generated and closed in the pro-$\mathbf{V}$ topology of F, then there is an expansion of its Stallings' graph such that $\mathrm{Aut}(\mathcal{S}(H)^+) \in \mathbf{V}$.

Thus $G \in \mathbf{V}$.

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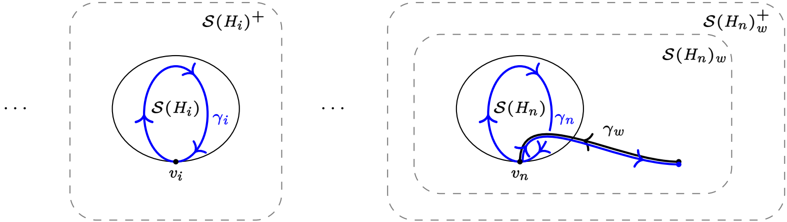
Let us see why. To show $w \in H_1 \cdots H_n$, it suffices to construct paths γ_i in Γ_i for $1 \leq i \leq n$ such that $[\gamma_1 \cdots \gamma_n]_F = 1$ and:



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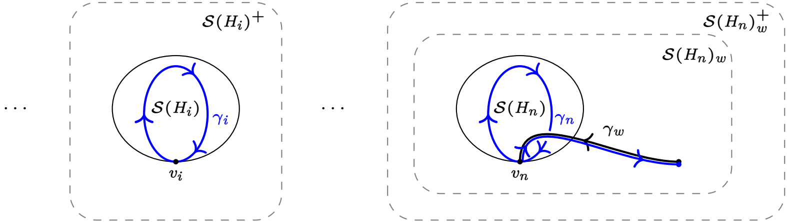
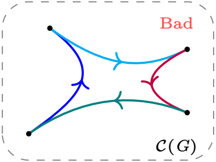
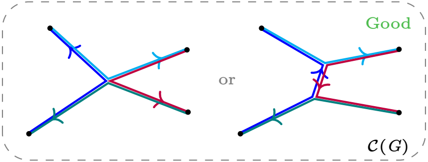


Paths $\gamma_1, \ldots, \gamma_n$ such that $[\gamma_1 \cdots \gamma_n]_G = 1$ are easy to obtain by since we assume $[w]_G \in [H_1 \cdots H_n]_G$; the issue is that we need $[\gamma_1 \cdots \gamma_n]_F = 1$.

Proof of the Ribes-Zalesskiĭ Theorem

The issue is that although $[\gamma_1 \cdots \gamma_n]_G = 1$, the paths that it traces out in the Cayley graph $\mathcal{C}(G)$ of G bounds a non-homotopically trivial loop.





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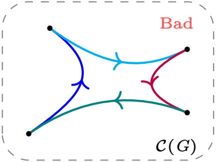

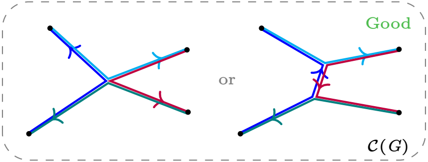

Lets strengthen the power of G to also keep track of how many times an edge is traversed, and not just the endpoint!

Proof of the Ribes-Zalesskiĭ Theorem

Fix a prime p and let E^+ be the set of positively-oriented edges in the Cayley graph of G. Let $C_pE^+ := (\mathbb{Z}/p\mathbb{Z})^{\oplus E^+}$ and let

$$G^{\mathbf{Ab}(p)} := \langle (e_x, x) : x \in X \rangle \leq C_pE^+ \rtimes G,$$

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where $e_x \in E^+$ is the edge $(1, x)$. There is a natural map $F \twoheadrightarrow G \twoheadrightarrow G^{\mathbf{Ab}(p)}$, and for any $w \in F$, we have

$$[w]_{G^{\mathbf{Ab}(p)}} = \left(\sum_{e \in E^+} [w(e)]_p e, [w]_G\right)$$

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Lemma

For any group $G \in \mathbf{V}$, there is a prime p such that $G^{\mathbf{Ab}(p)} \in \mathbf{V}$.

Proof of the Ribes-Zalesskiĭ Theorem

We claim that $K \coloneqq G^{\mathbf{Ab}(p)}$ for appropriate p works when $n = 2$; the general case requires an iterated extension and a (painful but easy) induction.

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Indeed, if $[w]_K \in [H_1 H_2]_K$, then there exist paths γ_i' in Γ_i, for $i = 1, 2$, such that $[\gamma_1' \gamma_2']_K = 1$ as before; in particular, $[\gamma_1' \gamma_2']_G = 1$.

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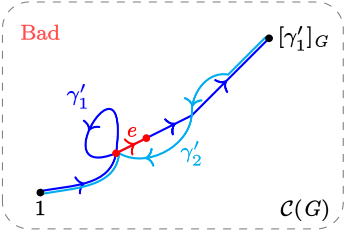
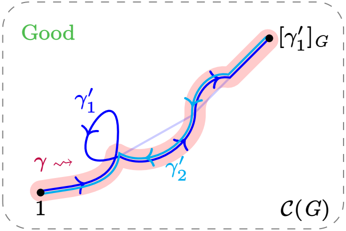
Indeed, if $[w]_K \in [H_1 H_2]_K$, then there exist paths γ_i' in Γ_i, for $i = 1, 2$, such that $[\gamma_1'\gamma_2']_K = 1$ as before; in particular, $[\gamma_1'\gamma_2']_G = 1$. Tracing these paths out in the Cayley graph of G gives the following picture.





■

Thank you!

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